The trace formula for transversally elliptic operators on Riemannian foliations

Yuri A. KORDYUKOV

Department of Mathematics,

Ufa State Aviation Technical University,

K.Marx str. 12, Ufa 450000, Russia

e-mail: yurikor@math.ugatu.ac.ru

September 1, 1999

1 Introduction

The main goal of the paper is to generalize the Duistermaat-Guillemin trace formula to the case of transversally elliptic operators on a compact foliated manifold. First, let us recall briefly the setting of the classical formula.

Let P be a positive self-adjoint elliptic pseudodifferential operator of order one on a closed manifold M (for example, $P=\sqrt{\Delta}$, where Δ is the Laplace-Beltrami operator of a Riemannian metric on M). For any function $f\in C^\infty_c(\mathbb{R})$, the operator $U_f=\int f(t)e^{itP}dt$ can be shown to be of trace class, and the mapping $\theta : f\mapsto \operatorname{tr} U_f$ is a continuous linear functional on $C^\infty_c(\mathbb{R})$. Otherwise speaking, θ is a distribution on $\mathbb{R}$. The principal symbol p of the operator P is a smooth function on the symplectic manifold $T^*M\setminus 0$, and, by definition, the bicharacteristic flow f_t of the operator P is the Hamiltonian flow on $T^*M\setminus 0$ defined by the function p.

By the theorem due to Colin de Verdiére and Chazarain [1, 2], the singularities of the distribution θ are contained in the period set of closed trajectories of the bicharacteristic flow f_t. Moreover, Duistermaat and Guillemin showed [3] that, under the assumption that the bicharacteristic flow is clean, one can write down an asymptotic expansion for the distribution θ near a given period of closed bicharacteristic. A formula for the leading term of this asymptotic expansion is the Duistermaat-Guillemin trace formula mentioned above. It involves the geometry of the bicharacteristic flow in the form of Poincaré map and Maslov indices and provides a far-reaching generalization of the classical Poisson formula and the Selberg trace formula on hyperbolic spaces.

In the paper, we prove a trace formula for an operator $P=\sqrt{A}$, where A is a positive self-adjoint transversally elliptic pseudodifferential operator of order two with the positive, holonomy invariant transverse principal symbol on a compact foliated manifold $(M,\mathcal{F})$ (see Theorem 6). One can consider such an operator as an elliptic operator on the singular

space $M/\mathcal{F}$ of leaves of the foliation $\mathcal{F}$ (this statement can be made more precise, using the language of noncommutative geometry, see [4]) the trace formula stated in this paper as an example of a trace formula for elliptic operators on singular spaces. We hope that this formula will be useful in further study of a general trace formula in noncommutative geometry (see, for instance, [5, 6] for discussion of a noncommutative trace formula).

It should be also noted that our trace formula can be viewed as a relative version of the Duistermaat-Guillemin trace formula.

2 Preliminaries and main results

Let $(M, \mathcal{F})$ be a compact, connected, oriented foliated manifold. We will use the following notation: $T\mathcal{F}$ is the tangent bundle, $H\mathcal{F} = TM/T\mathcal{F}$ is the normal bundle and $N^*\mathcal{F}$ is the conormal bundle to $\mathcal{F}$. There is a short exact sequence

$$0 \longrightarrow T\mathcal{F} \longrightarrow TM \longrightarrow H\mathcal{F} \longrightarrow 0. \tag{2.1}$$

We will consider linear operators, acting on half-densities. Recall that an α-density ($\alpha \in \mathbb{R}$) on a real vector space V of dimension n is a map $\phi : \Lambda^n V \to \mathbb{R}$ such that $\phi(\lambda v) = |\lambda|^\alpha \phi(v), v \in \Lambda^n V, \lambda \in \mathbb{R}$. For any real vector bundle E over a smooth manifold X, we will denote by $|E|^\alpha$ the α-density bundle of E.

Given a pseudodifferential operator $A \in \Psi^m(M, |TM|^{1/2})$, **the transversal principal symbol** σ_A of A is defined to be the restriction of its principal symbol a_m on $\tilde{N}^*\mathcal{F} = N^*\mathcal{F} \setminus 0$. An operator $A \in \Psi^m(M, |TM|^{1/2})$ is said to be **transversally elliptic**, if $\sigma_A(\nu) \neq 0$ for any $\nu \in \tilde{N}^*\mathcal{F}$.

For any smooth leafwise path γ from $x \in M$ to $y \in M$, sliding along leaves of the foliation defines **the holonomy map** h_γ, which associates to every germ of a local transversal to the foliation at the point x a germ of a local transversal to the foliation at the point y (this map is a natural generalization of the Poincaré first-return map for flows). The differential of this map (the linear holonomy map) is well-defined as a linear map $dh_\gamma : H_x\mathcal{F} \to H_y\mathcal{F}$ and the codifferential as a linear map $dh_\gamma^* : N_y^*\mathcal{F} \to N_x^*\mathcal{F}$.

The transversal principal symbol σ_A of an operator $A \in \Psi^m(M, |TM|^{1/2})$ is said to be **holonomy invariant**, if $\sigma_A(dh_\gamma^*(\nu)) = \sigma_A(\nu)$ for any smooth leafwise path γ from x to y and for any $\nu \in \tilde{N}_y^*\mathcal{F}$.

Throughout in the paper, we will assume that A is a linear operator in $C^\infty(M, |TM|^{1/2})$, satisfying the following conditions:

(A1) $A \in \Psi^2(M, |TM|^{1/2})$ is a transversally elliptic operator with the positive, holonomy invariant transversal principal symbol;

(A2) A is an essentially self-adjoint positive operator in L^2 space of half-densities on M, $L^2(M)$ (with the initial domain $C^\infty(M, |TM|^{1/2})$).

Example 1. A geometrical example of an operator, satisfying the conditions (A1) and (A2), is given by the operator $A = I + \Delta_H$, where Δ_H is the transversal Laplacian of a bundle-like metric on a Riemannian foliation.

Recall that a foliation $\mathcal{F}$ on a smooth Riemannian manifold (M, g_M) is **Riemannian** if it satisfies one of the following equivalent conditions (see, for instance, [7]):

1. $(M, \mathcal{F})$ locally has the structure of Riemannian submersion;
2. the transverse part of the Riemannian metric g_M (that is, its restriction to $H = T\mathcal{F}^\perp$) is holonomy invariant;
3. the horizontal distribution H is totally geodesic.

In this case, the metric g_M is called **bundle-like**.

The Riemannian metric g_M defines a decomposition of the cotangent bundle T^*M into a direct sum $T^*M = F^* \oplus H^*$. With respect to this decomposition, the de Rham differential $d : C^\infty(M) \to C^\infty(M, T^*M)$ can be written as a sum $d = d_F + d_H$, where $d_F : C^\infty(M) \to C^\infty(M, F^*)$ and $d_H : C^\infty(M) \to C^\infty(M, H^*)$.

The transversal Laplacian is a second order transversally elliptic differential operator in the space $C^\infty(M)$ defined by the formula

$$\Delta_H = -d_H^* d_H.$$

Its principal symbol a_2 is given by the formula

$$a_2(x, \xi) = g_H(\xi, \xi), \quad (x, \xi) \in \tilde{T}^*M,$$

and the holonomy invariance of the transverse principal symbol σ_{Δ_H} is equivalent to the assumption on the Riemannian metric g_M to be bundle-like.

From now on, we will assume that A satisfies the assumptions (A1) and (A2). By the spectral theorem, the operator $P = \sqrt{A}$ generates a strongly continuous group e^{itP} of bounded operators in $L^2(M)$. To define a distributional trace of the operator e^{itP}, one need an additional regularization. First, let us introduce some notation.

Recall that the holonomy groupoid $G = G_{\mathcal{F}}$ of the foliation $\mathcal{F}$ is the set of equivalence classes of leafwise paths $\gamma : [0, 1] \to M$ with respect to an equivalence relation $\sim_h$, setting $\gamma_1 \sim_h \gamma_2$ if γ_1 and γ_2 have the same initial and final points and the same holonomy maps. G is equipped with maps $s, r : G \to M$ given by $s(\gamma) = \gamma(0)$ and $r(\gamma) = \gamma(1)$ and has a composition law given by the composition of paths. For any $\gamma_1, \gamma_2 \in G$, the composition $\gamma_1 \circ \gamma_2$ makes sense iff $r(\gamma_2) = s(\gamma_1)$. We will make use of standard notation: $G^x = r^{-1}(x)$, $G_x = s^{-1}(x)$, $G_x^x = s^{-1}(x) \cap r^{-1}(x)$, $x \in M$. For any $x \in M$, G_x^x is the holonomy group of the leaf L_x through the point x and the maps $s : G^x \to L_x$ and $r : G_x \to L_x$ are covering maps associated with G_x^x. We will identify a point $x \in M$ with the element in G given by the constant path $\gamma(t) = x, t \in [0, 1]$.

Let $s^*(|T\mathcal{F}|^{1/2})$ and $r^*(|T\mathcal{F}|^{1/2})$ be the lifts of the vector bundle of leafwise half-densities $|T\mathcal{F}|^{1/2}$ to vector bundles on G via the mappings s and r respectively, and $|T\mathcal{G}|^{1/2} = r^*(|T\mathcal{F}|^{1/2}) \otimes s^*(|T\mathcal{F}|^{1/2})$. The line bundle $|T\mathcal{G}|^{1/2}$ is the bundle of leafwise half-densities on G with respect to the natural foliation $\mathcal{G}$ [8].

The space $C_c^\infty(G, |T\mathcal{G}|^{1/2})$ has the structure of involutive algebra (see, for instance, [8]). There is a natural $*$-representation R of the involutive algebra $C_c^\infty(G, |T\mathcal{G}|^{1/2})$ in $L^2(M)$.

For any $k \in C_c^\infty(G, |T\mathcal{G}|^{1/2})$, the operator $R(k)$ in $L^2(M)$ is defined as follows. According to the short exact sequence (2.1), the half-density vector bundle $|TM|^{1/2}$ can be decomposed as

$$|TM|^{1/2} \cong |T\mathcal{F}|^{1/2} \otimes |H\mathcal{F}|^{1/2}.$$

For any $\gamma \in G, s(\gamma) = x, r(\gamma) = y$, the corresponding linear holonomy map defines a map

$$dh_\gamma^* : |H_y\mathcal{F}|^{1/2} \to |H_x\mathcal{F}|^{1/2}.$$

Given $u \in L^2(M)$ of the form $u = u_1 \otimes u_2, u_1 \in L^2(M, |T\mathcal{F}|^{1/2}), u_2 \in L^2(M, |H\mathcal{F}|^{1/2})$, $R(k)u \in L^2(M)$ is defined by the formula

$$R(k)u(x) = \int_{G^x} k(\gamma) s^* u_1(\gamma) \otimes dh^*_{\gamma^{-1}}[u_2(s(\gamma))], \quad x \in M.$$

Proposition 2. *For any $k \in C_c^\infty(G, |T\mathcal{G}|^{1/2})$ and $f \in C_c^\infty(\mathbb{R})$, the operator $R(k) \int f(t) e^{itP} dt$ is of trace class. Moreover, for any $k \in C_c^\infty(G, |T\mathcal{G}|^{1/2})$, the formula*

$$\langle \theta_k, f \rangle = \operatorname{tr} R(k) \int f(t) e^{itP} dt, \quad f \in C_c^\infty(\mathbb{R}),$$

defines a distribution θ_k on the real line $\mathbb{R}$, $\theta_k \in \mathcal{D}'(\mathbb{R})$.

Let $\mathcal{F}_N$ be a foliation in $\tilde{N}^*\mathcal{F}$, which is the horizontal foliation for the natural leafwise flat connection in $\tilde{N}^*\mathcal{F}$ (the Bott connection). The leaf of the foliation $\mathcal{F}_N$ through a point $\nu \in \tilde{N}^*\mathcal{F}$ is the set of all $dh_\gamma^*(\nu) \in \tilde{N}^*\mathcal{F}$ such that $\gamma \in G, r(\gamma) = \pi(\nu)$, where $\pi : N^*\mathcal{F} \to M$ is the natural projection.

Denote by $H\mathcal{F}_N$ the normal bundle to $\mathcal{F}_N$, $H\mathcal{F}_N = T(N^*\mathcal{F})/T\mathcal{F}_N$. For any $\nu \in N^*\mathcal{F}$, the space $T_\nu N^*\mathcal{F}$ is a coisotropic subspace of the space $T_\nu T^*M$ equipped with the canonical symplectic structure, and $T_\nu\mathcal{F}_N$ its skew-orthogonal complement, therefore, the normal bundle $H_\nu\mathcal{F}_N$ has a natural symplectic structure (see, for instance, [9]).

Given an operator A under the conditions (A1) and (A2) with the principal symbol a, let $\tilde{p}$ be a smooth function on $\tilde{T}^*M$ homogeneous of degree one such that $\tilde{p}(\xi) \neq 0$ for $\xi \in \tilde{T}^*M$, which is equal to $p = a^{1/2}$ in some conical neighborhood of $N^*\mathcal{F}$, and $\tilde{f}_t$ the Hamiltonian flow of the function $\tilde{p}$. Define σ_P to be the restriction of p on $N^*\mathcal{F}$: $\sigma_P = \sigma_A^{1/2}$. The function σ_P coincides with the transverse principal symbol of any operator $P_1 \in \Psi^1(M, |TM|^{1/2})$ such that the principal symbols of P_1^2 and A are equal on $N^*\mathcal{F}$.

The holonomy invariance assumption on σ_A implies

$$d\tilde{p}(\nu)(X) = 0, \quad \nu \in \tilde{N}^*\mathcal{F}, \quad X \in T_\nu\mathcal{F}_N. \tag{2.2}$$

Using (2.2) and the fact that $\tilde{f}_t$ preserves the symplectic structure of T^*M, one can easily check that the Hamiltonian flow $\tilde{f}_t$ can be restricted on $N^*\mathcal{F}$. The resulting flow will be denoted by f_t. By definition, the flow f_t depends only on the 1-jet of the principal symbol a on $N^*\mathcal{F}$, therefore, it doesn't depend on a choice of $\tilde{p}$ and can be naturally called **the transverse bicharacteristic flow** of the operator A.

Since $\tilde{f}_t$ preserves the symplectic structure of T^*M and $T\mathcal{F}_N$ is the skew-adjoint complement to $TN^*\mathcal{F}$, f_t maps leaves of the foliation $\mathcal{F}_N$ to leaves. In particular, the differential df_t defines a map $T_\nu\mathcal{F}_N \to T_{f_t(\nu)}\mathcal{F}_N$ and a symplectic map $H_\nu\mathcal{F}_N \to H_{f_t(\nu)}\mathcal{F}_N$.

We say that a point $\nu \in \tilde{N}^*\mathcal{F}$ is **a relative fixed point** of the diffeomorphism $f_t : \tilde{N}^*\mathcal{F} \to \tilde{N}^*\mathcal{F}$ (with respect to the foliation $\mathcal{F}_N$), if there exist $\gamma \in G$ such that $r(\gamma) = \pi(\nu)$ and $f_{-t}\ dh^*_\gamma(\nu) = \nu$.

For any $t \in \mathbb{R}$, denote by Z_t the set of relative fixed points of f_t. We also introduce the corresponding set in the cospherical bundle $SN^*\mathcal{F} = \{\nu \in N^*\mathcal{F} : \sigma_P(\nu) = 1\}$: $SZ_t = Z_t \cap SN^*\mathcal{F}$. This set might be not closed, but, for any $k \in C^\infty_c(G, |T\mathcal{G}|^{1/2}))$, the corresponding part

$$SZ_{t,k} = \{\nu \in SN^*\mathcal{F} : (\exists \gamma \in \operatorname{supp} k, r(\gamma) = \pi(\nu)) f_{-t}\ dh^*_\gamma(\nu) = \nu\}$$

is closed. By the tranversal ellipticity of σ_A, the flow f_t is transverse to $\mathcal{F}_N$, therefore, **the relative period set** $\mathcal{T}_k = \{t \in \mathbb{R} : SZ_{t,k} \neq \varnothing\}$ is a discrete subset of $\mathbb{R}$.

The following theorem was proved in [10], but we will give an independent proof.

Theorem 3. *Given an operator A under the conditions (A1) and (A2) and $k \in C^\infty_c(G, |T\mathcal{G}|^{1/2})$, the distribution θ_k is smooth outside of the relative period set $\mathcal{T}_k$ of the transverse bicharacterictic flow f_t.*

Let $G_{\mathcal{F}_N}$ denote the holonomy groupoid of the foliation $\mathcal{F}_N$. $G_{\mathcal{F}_N}$ consists of all pairs $(\gamma, \nu) \in G_\mathcal{F} \times \tilde{N}^*\mathcal{F}$ such that $r(\gamma) = \pi(\nu)$ with the source map $s_N : G_{\mathcal{F}_N} \to \tilde{N}^*\mathcal{F}, s_N(\gamma, \nu) = dh^*_\gamma(\nu)$, and the target map $r_N : G_{\mathcal{F}_N} \to \tilde{N}^*\mathcal{F}, r_N(\gamma, \nu) = \nu$. There is a projection $\pi_G : G_{\mathcal{F}_N} \to G_\mathcal{F}$ given by the formula $\pi_G(\gamma, \nu) = \gamma$. Put also $G_{SN^*\mathcal{F}} = G_{\mathcal{F}_N} \cap r_N^{-1}(SN^*\mathcal{F})$.

For any $(\gamma, \nu) \in G_{\mathcal{F}_N}$, denote by $dH_{(\gamma,\nu)}$ the associated linear holonomy map:

$$dH_{(\gamma,\nu)} : H_{dh^*_\gamma(\nu)}\mathcal{F}_N \to H_\nu\mathcal{F}_N.$$

It is easy to see that $dH_{(\gamma,\nu)}$ preserves the symplectic structure of $H\mathcal{F}_N$.

Denote by $Q : TN^*\mathcal{F} \to H\mathcal{F}_N$ the projection map. The differential of the map $(r_N, s_N) : G_{\mathcal{F}_N} \to \tilde{N}^*\mathcal{F} \times \tilde{N}^*\mathcal{F}$ defines an isomorphism of the tangent space $T_{(\gamma,\nu)}G_{\mathcal{F}_N}$ with the set of all $(v_1, v_2) \in T_\nu N^*\mathcal{F} \oplus T_{dh^*_\gamma(\nu)}N^*\mathcal{F}$ such that the normal components of v_1 and v_2 are connected by the holonomy map: $Q(v_1) = dH_{(\gamma,\nu)}(Q(v_2))$.

The holonomy groupoid $G_{\mathcal{F}_N}$ has the natural foliation $\mathcal{G}_{\mathcal{F}_N}$ such that the tangent bundle $T_{(\gamma,\nu)}\mathcal{G}_{\mathcal{F}_N}$ corresponds to $T_\nu\mathcal{F}_N \oplus T_{dh^*_\gamma(\nu)}\mathcal{F}_N$ under the isomorphism described above. The normal space $H_{(\gamma,\nu)}\mathcal{G}_{\mathcal{F}_N}$ to $\mathcal{G}_{\mathcal{F}_N}$ is isomorphic to the set of all $(v_1, v_2) \in H_\nu\mathcal{F}_N \oplus H_{dh^*_\gamma(\nu)}\mathcal{F}_N$ such that $v_1 = dH_{(\gamma,\nu)}(v_2)$, and therefore the maps dr_N (ds_N) define isomorphisms of $H_{(\gamma,\nu)}\mathcal{G}_{\mathcal{F}_N}$ with $H_\nu\mathcal{F}_N$ ($H_{dh^*_\gamma(\nu)}\mathcal{F}_N$) respectively.

Lemma 4. *The set Z_t is a saturated subset of $N^*\mathcal{F}$, that is, it is a union of leaves of the foliation $\mathcal{F}_N$.*

Proof. By the holonomy invariance of p, the Hamiltonian vector field Ξ_p with the Hamiltonian p satisfies the identity

$$dH_{(\gamma,\nu)}(Q(\Xi_p(dh^*_\gamma(\nu)))) = Q(\Xi_p(\nu)), \quad \nu \in \tilde{N}^*\mathcal{F}, \quad \gamma \in G, \quad r(\gamma) = \pi(\nu),$$

therefore, there exists a vector field $\hat{\Xi}_p$ on $G_{\mathcal{F}_N}$ such that

$$ds_N(\hat{\Xi}_p(\gamma,\nu)) = \Xi_p(dh^*_\gamma(\nu)), \quad dr_N(\hat{\Xi}_p(\gamma,\nu)) = \Xi_p(\nu), \quad (\gamma,\nu)\in G_{\mathcal{F}_N}. \tag{2.3}$$

Let $\hat{F}_t$ be a flow on $G_{\mathcal{F}_N}$ generated by $\hat{\Xi}_p$. By (2.3), we have

$$f_t\circ r_N = r_N\circ \hat{F}_t, \quad f_t\circ s_N = s_N\circ \hat{F}_t,$$

or, if we write $\hat{F}_t : G_{\mathcal{F}_N}\to G_{\mathcal{F}_N}$ as $\hat{F}_t(\gamma,\nu) = (F_t(\gamma,\nu), f_t(\nu))$,

$$f_t(dh^*_\gamma(\nu)) = dh^*_{F_t(\gamma,\nu)}(f_t(\nu)). \tag{2.4}$$

Take any $\nu\in Z_t$ with the corresponding $\gamma\in G$ such $r(\gamma)=\pi(\nu), f_{-t}\ dh^*_\gamma(\nu)=\nu$. Let $(\gamma_1,\nu)\in G_{\mathcal{F}_N}$. Then we have

$$\begin{aligned} f_t(dh^*_{\gamma_1}(\nu)) &= dh^*_{F_t(\gamma_1,\nu)}(f_t(\nu)) \qquad \text{by (2.4)}\\ &= dh^*_{F_t(\gamma_1,\nu)}(dh^*_\gamma(\nu))\\ &= (dh^*_{F_t(\gamma_1,\nu)}\circ dh^*_\gamma\circ dh^*_{\gamma_1^{-1}})(dh^*_{\gamma_1}(\nu))\\ &= dh^*_{\gamma'}(dh^*_{\gamma_1}(\nu)),\end{aligned}$$

where $\gamma'=\gamma_1^{-1}\circ\gamma\circ F_t(\gamma_1,\nu)$ that implies $dh^*_{\gamma_1}(\nu)\in Z_t$. □

The relative fixed point sets Z_t can be naturally lifted to the holonomy groupoid $G_{\mathcal{F}_N}$:

$$\mathcal{Z}_t = \{(\gamma,\nu)\in G_{\mathcal{F}_N} : f_{-t}\ dh^*_\gamma(\nu)=\nu\}, \quad S\mathcal{Z}_t = \mathcal{Z}_t\cap G_{SN^*\mathcal{F}},$$

By Lemma 4, $Z_t = r_N(\mathcal{Z}_t) = s_N(\mathcal{Z}_t)$.

Let us assume that $\mathcal{Z}_t$ is a smooth submanifold of $G_{\mathcal{F}_N}$. By Lemma 4, the tangent space to $\mathcal{Z}_t$ at a point $(\gamma,\nu)\in\mathcal{Z}_t$ contains a subspace $F_{(\gamma,\nu)}\mathcal{Z}_t$, which is the graph of the linear map $df_t(\nu) : T_\nu\mathcal{F}_N\to T_{dh^*_\gamma(\nu)}\mathcal{F}_N = T_{f_t(\nu)}\mathcal{F}_N$:

$$F_{(\gamma,\nu)}\mathcal{Z}_t = \{(v_1,v_2)\in T_\nu\mathcal{F}_N\times T_{dh^*_\gamma(\nu)}\mathcal{F}_N : v_2 = df_t(\nu)(v_1)\}.$$

Let

$$H_{(\gamma,\nu)}\mathcal{Z}_t = T_{(\gamma,\nu)}\mathcal{Z}_t/F_{(\gamma,\nu)}\mathcal{Z}_t, \quad H_{(\gamma,\nu)}S\mathcal{Z}_t = T_{(\gamma,\nu)}S\mathcal{Z}_t/F_{(\gamma,\nu)}\mathcal{Z}_t.$$

Definition 5. Let $t\in\mathbb{R}$ be a relative period of the flow f_t. We say that the flow f_t is **clean** on $\mathcal{Z}_t$, if:

(1) $\mathcal{Z}_t$ is a smooth submanifold of $G_{\mathcal{F}_N}$;

(2) the normal space $H_{(\gamma,\nu)}\mathcal{Z}_t$ at any point $(\gamma,\nu)\in\mathcal{Z}_t$ coincides with the set of all $(v_1,v_2)\in H_{(\gamma,\nu)}\mathcal{G}_{\mathcal{F}_N}$ such that $v_2 = df_t(\nu)(v_1)$.

Let $|T\mathcal{F}_N|^{1/2}$ be the vector bundle of leafwise half-densities on $N^*\mathcal{F}$, and $s_N^*(|T\mathcal{F}_N|^{1/2})$ and $r_N^*(|T\mathcal{F}_N|^{1/2})$ are the lifts of this vector bundle to vector bundles on $G_{\mathcal{F}_N}$ via the mappings s_N and r_N respectively. Let $|T\mathcal{G}_{\mathcal{F}_N}|^{1/2}$ be the vector bundle of leafwise half-densities on $G_{\mathcal{F}_N}$:

$$|T\mathcal{G}_{\mathcal{F}_N}|^{1/2} = r_N^*(|T\mathcal{F}_N|^{1/2}) \otimes s_N^*(|T\mathcal{F}_N|^{1/2}).$$

The projection $\pi_G : G_{\mathcal{F}_N} \to G$ defines a local diffeomorphism $\pi_G : \mathcal{G}_{\mathcal{F}_N} \to \mathcal{G}$, that induces a map

$$\pi_G^* : C_c^\infty(G, |T\mathcal{G}|^{1/2}) \to C^\infty(G_{\mathcal{F}_N}, |T\mathcal{G}_{\mathcal{F}_N}|^{1/2}).$$

Define a restriction map

$$R_{\mathcal{Z}} : C_c^\infty(G_{\mathcal{F}_N}, |T\mathcal{G}_{\mathcal{F}_N}|^{1/2}) \to C_c^\infty(\mathcal{Z}_t, |T\mathcal{F}_N|^1)$$

as follows. If $\rho = f r_N^*\rho_1 \otimes s_N^*\rho_2, \quad f \in C_c^\infty(G_{\mathcal{F}_N}), \quad \rho_1, \rho_2 \in C_c^\infty(M, |T\mathcal{F}_N|^{1/2})$, then

$$R_{\mathcal{Z}}\rho(\gamma,\nu) = f(\gamma,\nu)\rho_1(\gamma,\nu) df_t^*(\nu)[\rho_2(dh_\gamma^*(\nu))], \quad (\gamma,\nu) \in \mathcal{Z}_t,$$

where the map $df_t^*(\nu) : |T_{f_t(\nu)}\mathcal{F}_N|^{1/2} \to |T_\nu\mathcal{F}_N|^{1/2}$ is induced by the linear map $df_t(\nu) : T_\nu\mathcal{F}_N \to T_{f_t(\nu)}\mathcal{F}_N$.

If the flow f_t is clean, there is defined a natural density $d\mu_{\mathcal{Z}}$ on $H_{(\gamma,\nu)}\mathcal{Z}_t$, being the fixed point set of the symplectic linear map $dH_{(\gamma,\nu)} \circ df_t(\nu)$ of the symplectic space $H_\nu\mathcal{F}_N$ (see, for instance, [3, Lemma 4.3]). Dividing $d\mu_{\mathcal{Z}}$ by $d\sigma_P$, we get a density $d\mu_{S\mathcal{Z}}$ on $H_{(\gamma,\nu)}S\mathcal{Z}_t$.

Using the natural isomorphism

$$|TS\mathcal{Z}_t| \cong |F\mathcal{Z}| \otimes |HS\mathcal{Z}_t|.$$

one can combine the densities $R_{\mathcal{Z}}\pi_G^*k \in C_c^\infty(S\mathcal{Z}_t, |FS\mathcal{Z}_t|)$ and $d\mu_{S\mathcal{Z}} \in C_c^\infty(S\mathcal{Z}_t, |HS\mathcal{Z}_t|)$ to get a smooth density $R_{\mathcal{Z}}\pi_G^*k\, d\mu_{S\mathcal{Z}}$ on $S\mathcal{Z}_t$.

Let $\sigma_{\rm sub}(A)$ denote the subprincipal symbol of A. Define $\sigma_{\rm sub}(P) = \frac{1}{2}a^{-\frac{1}{2}}\sigma_{\rm sub}(A)$ in some conic neighborhood of $N^*\mathcal{F}$. The restriction of $\sigma_{\rm sub}(P)$ on $N^*\mathcal{F}$ is equal to the restriction on $N^*\mathcal{F}$ of the subprincipal symbol of any operator $P_1 \in \Psi^1(M, |TM|^{1/2})$ such that the complete symbols of P_1^2 and A are equal $\mod S^{-\infty}$ in some neighborhood of $N^*\mathcal{F}$.

Theorem 6. *Let $t \in \mathbb{R}$ be a relative period of the flow f_t. Assume that the relative fixed point set $\mathcal{Z}_t$ is clean. Then, for any $k \in C_c^\infty(G, |T\mathcal{G}|^{1/2})$ and for any τ in some neighborhood of t, we have*

$$\theta_k(\tau) = \sum_{\mathcal{Z}_j} \int_{-\infty}^{+\infty} \alpha_j(s,k) e^{is(\tau-t)} ds, \tag{2.5}$$

where:

1. *$\mathcal{Z}_j$ are all connected components of the set $S\mathcal{Z}_t$ in $G_{SN^*\mathcal{F}}$ of dimensions $d_j = \dim \mathcal{Z}_j$;*

2. α_j has an asymptotic expansion

$$\alpha_j(s,k) \sim \left(\frac{s}{2\pi i}\right)^{(d_j-p-1)/2} i^{-\sigma_j} \sum_{r=0}^{+\infty} \alpha_{j,r}(k) s^{-r}, \quad s \to +\infty \tag{2.6}$$

with $\alpha_{j,0}$ given by the formula

$$\alpha_{j,0}(k) = \int_{\mathcal{Z}_j} e^{i\int_0^t \sigma_{\text{sub}}(P)(f_{-\tau} dh_\gamma^*(\nu))d\tau} R_{\mathcal{Z}} \pi_G^* k(\gamma,\nu) d\mu_{S\mathcal{Z}_j}(\gamma,\nu), \tag{2.7}$$

where σ_j denotes the Maslov index associated with the connected component $\mathcal{Z}_j$ (see below for the definition).

3 Reduction to the case when A is elliptic

In this section, we will assume that A is an operator under the assumptions (A1) and (A2). We will use the classes $\Psi^{m,-\infty}(M,\mathcal{F},|TM|^{1/2})$ of transversal pseudodifferential operators (see [4] for the definition) and the Sobolev spaces $H^s(M)$ of half-densities on M. Put also $\Psi^{*,-\infty}(M,\mathcal{F},|TM|^{1/2}) = \bigcup_m \Psi^{m,-\infty}(M,\mathcal{F},|TM|^{1/2})$.

By [4], the operator $P = A^{1/2}$ satisfies the following conditions:

(H1) P has the form

$$P = P_1 + R_1,$$

where:

(a) $P_1 \in \Psi^1(M,|TM|^{1/2})$ is a transversally elliptic operator with the positive, holonomy invariant transversal principal symbol such that the complete symbols of P_1^2 and A are equal mod $S^{-\infty}$ in some neighborhood of $N^*\mathcal{F}$;

(b) R_1 is a bounded operator from $L^2(M)$ to $H^{-1}(M)$ and for any $K \in \Psi^{*,-\infty}(M,\mathcal{F},|TM|^{1/2})$ the operator KR_1 is a smoothing operator in $L^2(M)$, that is, it defines a bounded operator from $L^2(M)$ to $C^\infty(M,|TM|^{1/2})$.

(H2) P is essentially self-adjoint in $L^2(M)$ (with the initial domain $C^\infty(M,|TM|^{1/2})$).

Lemma 7 ([10]). *Any operator P, satisfying the conditions (H1) and (H2), can be represented in the form*

$$P = P_2 + R_2, \tag{3.1}$$

where:

(a) $P_2 \in \Psi^1(M,|TM|^{1/2})$ is an essentially self-adjoint, elliptic operator with the positive principal symbol and the holonomy invariant transversal principal symbol such that the complete symbols of P_1 and P_2 are equal mod $S^{-\infty}$ *in some neighborhood of $N^*\mathcal{F}$;*

(b) R_2 is a bounded operator from $L^2(M)$ to $H^{-1}(M)$ and, for any $K \in \Psi^{*,-\infty}(M,\mathcal{F},|TM|^{1/2})$, the operator KR_2 is a smoothing operator in $L^2(M)$.

Proof. Take a foliated coordinate chart Ω on M with coordinates $(x, y) \in I^p \times I^q$ (I is the open interval $(0, 1)$) such that the restriction of $\mathcal{F}$ on U is given by the sets $y = \text{const}$. Let $p_1 \in S^1(I^n \times \mathbb{R}^n)$ be the complete symbol of the operator P_1 in this chart. Assume that $p_1(x, y, \xi, \eta)$ is invertible for any $(x, y, \xi, \eta) \in U, |\xi|^2 + |\eta|^2 > R^2$, where $R > 0$, U is a conic neighborhood of the set $\eta = 0$. Take any function $\phi \in C^\infty(I^n \times \mathbb{R}^n), \phi = \phi(x, y, \xi, \eta), x \in I^p, y \in I^q, \xi \in \mathbb{R}^p, \eta \in \mathbb{R}^q$, homogeneous of degree 0 in (ξ, η) for $|\xi|^2 + |\eta|^2 > 1$, which is supported in some conic neighborhood of $\eta = 0$ and is equal to 1 in U, and put

$$p_2(x, y, \xi, \eta) = \phi p_1(x, y, \xi, \eta) + (1 - \phi)(1 + |\xi|^2 + |\eta|^2)^{1/2}.$$

Take P_2 to be the operator $p_2(x, y, D_x, D_y)$ with the complete symbol p_2 (or, more precisely, $p_2(x, y, D_x, D_y) + p_2(x, y, D_x, D_y)^*$ to provide self-adjointness) and put $R_2 = P - P_2$. The operator $P_1 - P_2$ has order $-\infty$ in some conic neighbourhood of $N^*\mathcal{F}$, therefore, for any $K \in \Psi^{*,-\infty}(M, \mathcal{F}, |TM|^{1/2})$ the operator $K(P_1 - P_2)$ is a smoothing operator [4], that completes immediately the proof. ◻

Denote by $W(t) = e^{itP_2}$ the wave group generated by the elliptic operator P_2. It is well-known that $W(t)$ is a Fourier integral operator (see below for more details). Put also $R(t) = e^{itP} - W(t)$.

Proposition 8. *For any $K \in \Psi^{*,-\infty}(M, \mathcal{F}, |TM|^{1/2})$, the family $KR(t), t \in \mathbb{R}$, is a smooth family of bounded operators from $L^2(M)$ to $C^\infty(M, |TM|^{1/2})$.*

Proof. Since $P^2 = A \in \Psi^2(M, |TM|^{1/2})$, by interpolation and duality, P defines a bounded operator from $H^1(M)$ to $L^2(M)$ and from $L^2(M)$ to $H^{-1}(M)$ and, for any natural N, P^N defines a bounded operator from $H^N(M)$ to $L^2(M)$ and from $L^2(M)$ to $H^{-N}(M)$. Since $R_2 = P - P_2$ and $P_2 \in \Psi^1(M, |TM|^{1/2})$, R_2 also defines a bounded operator from $H^1(M)$ to $L^2(M)$ and from $L^2(M)$ to $H^{-1}(M)$.

By assumption, for any $K \in \Psi^{*,-\infty}(M, \mathcal{F}, |TM|^{1/2})$, the operator KR_2 is a smoothing operator, therefore, the operator KR_2P^N is defined as an operator from $H^N(M)$ to $C^\infty(M, |TM|^{1/2})$.

Lemma 9. *For any $K \in \Psi^{*,-\infty}(M, \mathcal{F}, |TM|^{1/2})$ and for any $N \in \mathbb{N}$, the operator KR_2P^N extends to a bounded operator from $L^2(M)$ to $C^\infty(M, |TM|^{1/2})$.*

Proof. We will prove the lemma by induction on N. For $N = 0$, the statement is true by assumption. Let us assume that it is true for some N, that is, for any $K \in \Psi^{*,-\infty}(M, \mathcal{F}, |TM|^{1/2})$, the operator KR_2P^N extends to a bounded operator from $L^2(M)$ to $C^\infty(M, |TM|^{1/2})$.

We have the equality $P^2 - P_2^2 = R_2P + P_2R_2$ as operators from $H^1(M)$ to $H^{-1}(M)$, therefore, for any $K \in \Psi^{*,-\infty}(M, \mathcal{F}, |TM|^{1/2})$,

$$KR_2P^{N+1} = K(R_2P)P^N = K(P^2 - P_2^2)P^N - KP_2R_2P^N.$$

The operator $P^2 - P_2^2 \in \Psi^2(M, |TM|^{1/2})$ has order $-\infty$ in some conic neighbourhood of $N^*\mathcal{F}$, therefore, for any $K \in \Psi^{*,-\infty}(M, \mathcal{F}, |TM|^{1/2})$ the operator $K(P^2 - P_2^2)$ extends to a bounded operator from $H^s(M)$ to $C^\infty(M, |TM|^{1/2})$ for any s and $K(P^2 - P_2^2)P^N$ extends to a bounded operator from $L^2(M)$ to $C^\infty(M, |TM|^{1/2})$.

Since $P_2 \in \Psi^1(M,|TM|^{1/2})$ and $K \in \Psi^{*,-\infty}(M,\mathcal{F},|TM|^{1/2})$, by the composition theorem [4], $KP_2 \in \Psi^{*,-\infty}(M,\mathcal{F},|TM|^{1/2})$ and, by induction hypothesis, $KP_2R_2P^N$ extends to a bounded operator from $L^2(M)$ to $C^\infty(M,|TM|^{1/2})$. $\square$

By the Duhamel formula, we have

$$R(t)u = i\int_0^t e^{i\tau P_2}\, R_2\, e^{i(t-\tau)P} u\, d\tau, \quad u \in H^1(M) \subset D(P),$$

therefore, for any $K \in \Psi^{*,-\infty}(M,\mathcal{F},|TM|^{1/2})$,

$$KR(t) = i\int_0^t Ke^{i\tau P_2}\, R_2\, e^{i(t-\tau)P}\, d\tau = i\int_0^t e^{i\tau P_2}e^{-i\tau P_2}Ke^{i\tau P_2}\, R_2\, e^{i(t-\tau)P}\, d\tau.$$

Any operator $K \in \Psi^{*,-\infty}(M,\mathcal{F},|TM|^{1/2})$ is a Fourier integral operator (see below for more details) and, using the composition theorem for Fourier integral operators, one can check that $e^{-i\tau P_2}Ke^{i\tau P_2} \in \Psi^{*,-\infty}(M,\mathcal{F},|TM|^{1/2})$. Therefore, the operator $e^{-i\tau P_2}Ke^{i\tau P_2}R_2$ extends to a bounded operator from $L^2(M)$ to $C^\infty(M,|TM|^{1/2})$. Since $e^{i\tau P_2}$ maps $C^\infty(M,|TM|^{1/2})$ to $C^\infty(M,|TM|^{1/2})$ and, by the spectral theorem, $e^{i(t-\tau)P}$ is a bounded operator in $L^2(M)$, for any $K \in \Psi^{*,-\infty}(M,\mathcal{F},|TM|^{1/2})$, the operator $KR(t)$ extends to a bounded operator from $L^2(M)$ to $C^\infty(M,|TM|^{1/2})$. Moreover, one can be easily seen from above arguments that the function $KR(t)$ is continuous as a function on $\mathbb{R}$ with values in the space $\mathcal{L}(L^2(M),C^\infty(M,|TM|^{1/2}))$ of bounded operators from $L^2(M)$ to $C^\infty(M,|TM|^{1/2})$.

For any $u \in H^1(M)$, the function $\mathbb{R} \to H : t \mapsto KR(t)u$ is differentiable, and

$$\frac{d}{dt}KR(t)u = iK(Pe^{itP}u - P_2e^{itP_2}u) = i(KP_2R(t) + KR_2e^{itP})u.$$

The operator $KP_2R(t)+KR_2e^{itP}$ extends to a bounded operator from $L^2(M)$ to $C^\infty(M,|TM|^{1/2})$, and, moreover, the function $t \mapsto KP_2R(t) + KR_2e^{itP}$ is a continuous function on $\mathbb{R}$ with values in $\mathcal{L}(L^2(M),C^\infty(M,|TM|^{1/2}))$. Using this, one can be easily seen that the function $t \mapsto KR(t)$ is differentiable as a function on $\mathbb{R}$ with values in $\mathcal{L}(L^2(M),C^\infty(M,|TM|^{1/2}))$ and

$$\frac{d}{dt}KR(t) = i(KP_2R(t) + KR_2e^{itP}).$$

Let us proceed by induction. Assume that, for any $K \in \Psi^{*,-\infty}(M,\mathcal{F},|TM|^{1/2})$ and for some natural n, the function $KR(t)$ is n-times differentiable as a function on $\mathbb{R}$ with values in $\mathcal{L}(L^2(M),C^\infty(M,|TM|^{1/2}))$ and the derivative $KR^{(n)}(t), t \in \mathbb{R}$ satisfies the equation

$$KR^{(n)}(t) = iKP_2R^{(n-1)}(t) + i^nKR_2P^{n-1}e^{itP}. \tag{3.2}$$

To prove that the function $t \mapsto KR^{(n)}(t)$ is differentiable as a function on $\mathbb{R}$ with values in $\mathcal{L}(L^2(M),C^\infty(M,|TM|^{1/2}))$, as above, it suffices to prove that the derivative $(d/dt)KR^{(n)}(t)u$ exists for any u from a dense subspace of $L^2(M)$, it extends to a bounded operator from $L^2(M)$ to $C^\infty(M,|TM|^{1/2})$, and its extension is continuous as a function on $\mathbb{R}$ with values in $\mathcal{L}(L^2(M),C^\infty(M,|TM|^{1/2}))$.

From (3.2), one can easy to see that the derivative $(d/dt)KR^{(n)}(t)u$ exists for any $u \in H^1(M)$ and satisfies the equation

$$\frac{d}{dt}KR^{(n)}(t)u = iKP_2R^{(n)}(t)u + i^{n+1}KR_2P^ne^{itP}u. \tag{3.3}$$

The first term in the right-hand side of (3.3), $iKP_2R^{(n)}(t)$, is a bounded operator from $L^2(M)$ to $C^\infty(M, |TM|^{1/2})$ by the induction hypothesis. By Lemma 9, the operator KR_2P^n extends to a bounded operator from $L^2(M)$ to $C^\infty(M, |TM|^{1/2})$, and, by the spectral theorem, e^{itP} is a bounded operator in $L^2(M)$, therefore, the second term in the right-hand side of (3.3), the operator $i^{n+1}KR_2P^ne^{itP}$, extends to a bounded operator from $L^2(M)$ to $C^\infty(M, |TM|^{1/2})$. It is also clear the right-hand side of (3.3) is continuous as a function on $\mathbb{R}$ with values in $\mathcal{L}(L^2(M), C^\infty(M, |TM|^{1/2}))$. This completes the proof of the existence of the derivative $KR^{(n+1)}(t) = (d/dt)KR^{(n)}(t)$ and the induction arguments. □

Proof of Proposition 2. Let $W(t)$ and $R(t)$ be as in Proposition 8 and $k \in C_c^\infty(G, |T\mathcal{G}|^{1/2})$. Define $\theta_k(t)$ by the formula

$$\theta_k(t) = \operatorname{tr} R(k)W(t) + \operatorname{tr} R(k)R(t).$$

Since P_2 is an elliptic operator, the operator $\int f(t)e^{itP_2}dt$ is a smoothing operator in $\mathcal{D}'(M)$ and the trace of the operator $R(k)W(t)$ is well-defined as a distribution on $\mathbb{R}$ [3]. Since any bounded operator T in $L^2(M)$, which extends to a bounded operator from $L^2(M)$ to $H^s(M)$ with $s > n = \dim M$, is a trace class operator, the trace $\operatorname{tr} R(k)R(t)$ is a well-defined smooth function on $\mathbb{R}$ by Proposition 8. □

Corollary 10. *For any $k \in C_c^\infty(G, |T\mathcal{G}|^{1/2})$, the function $\operatorname{tr} R(k)R(t) = \theta_k(t) - \operatorname{tr} R(k)W(t)$ is a smooth function on $\mathbb{R}$.*

It should be noted that, without any additional assumption about the operator P in question, the corresponding distribution on $G_{\mathcal{F}_N} \times \mathbb{R}$, $k \mapsto \operatorname{tr} R(k)R(t)$, might be very singular, but the singularities of the distribution $k \mapsto \operatorname{tr} R(k)W(t)$ can be described rather explicitly under the clear intersection assumption.

4 The case of an elliptic operator

Let $P_2 \in \Psi^1(M, |TM|^{1/2})$ be an essentially self-adjoint, elliptic operator with the positive principal symbol and the holonomy invariant transversal principal symbol and $W(t) = e^{itP_2}$. The singularities of the distribution $t \mapsto \operatorname{tr} R(k)W(t)$ can be studied in a standard manner, using microlocal analysis.

Fix $k \in C_c^\infty(G, |T\mathcal{G}|^{1/2})$. We will consider the operator family $R(k)W(t)$ as a single operator $R(k)W$ from $L^2(M)$ to $L^2(\mathbb{R} \times M)$. We will prove that this operator is a Fourier integral operator. At first, let us recall well-known facts about the structure of the operators $R(k)$ and W.

As above, let $\tilde{p}$ be a smooth function on $\tilde{T}^*M$ homogeneous of degree one such that $\tilde{p}(\xi) \neq 0$ for $\xi \in \tilde{T}^*M$, which is equal to $p = a^{1/2}$ in some conic neighborhood of $N^*\mathcal{F}$ and $\tilde{f}_t$ the Hamiltonian flow of $\tilde{p}$. Without loss of generality, we may assume that $\tilde{p}$ is the principal symbol of the operator P_2. Let $\Lambda_{\tilde{p}}$ be the Lagrangian submanifold in $\tilde{T}^*\mathbb{R} \times \tilde{T}^*M \times \tilde{T}^*M$:

$$\Lambda_{\tilde{p}} = \{((t,\tau),(x,\xi),(y,\eta)) \in \tilde{T}^*\mathbb{R} \times \tilde{T}^*M \times \tilde{T}^*M : \tau = \tilde{p}(x,\xi), (x,\xi) = \tilde{f}_{-t}(y,\eta)\}.$$

Then W is a Fourier integral operator associated with the canonical relation $\Lambda'_{\tilde{p}}$, $W \in I^{-1/4}(\mathbb{R} \times M \times M, \Lambda'_{\tilde{p}})$.

The operator $R(k)$ belongs to $\Psi^{0,-\infty}(M, \mathcal{F}, |TM|^{1/2})$ and, therefore, is a Fourier integral operator associated with an immersed canonical relation, which is the image of $G_{\mathcal{F}_N}$ under the mapping

$$(r_N, s_N) : G_{\mathcal{F}_N} \to T^*M \times T^*M, \quad (\gamma, \nu) \mapsto (\nu, dh^*_\gamma(\nu)),$$

given by the source and the target mappings of the groupoid $G_{\mathcal{F}_N}$ (see [4]). More precisely, $R(k) \in I^{-p/2}(M \times M, G'_{\mathcal{F}_N})$.

By the tranversal ellipticity of $\tilde{p}$, the intersection of $\Lambda'_{\tilde{p}}$ with $G'_{\mathcal{F}_N}$ is transverse, and, by the composition theorem of Fourier integral operators [11], the operator $R(k)W$ is a Fourier integral operator associated with an immersed canonical relation from T^*M to $T^*(\mathbb{R} \times M)$ given by the map

$$\Pi : \mathbb{R} \times G_{\mathcal{F}_N} \to T^*\mathbb{R} \times T^*M \times T^*M, \quad (t, \gamma, \nu) \mapsto (t, p(\nu), \nu, f_{-t} dh^*_\gamma(\nu)).$$

More precisely, $R(k)W \in I^{-p/2-1/4}(\mathbb{R} \times M \times M; \mathbb{R} \times G_{\mathcal{F}_N}, \Pi)$.

Recall that the trace functional can be treated from the point of microlocal analysis as follows [3]. Let $\Delta : \mathbb{R} \times M \to \mathbb{R} \times M \times M$ be the diagonal map, $\Delta(t,x) = (t,x,x), (t,x) \in \mathbb{R} \times M$, and $\pi : \mathbb{R} \times M \to M$ the projection map. Then

$$\mathrm{tr}\ R(k)W = \pi_* \Delta^* W_k, \tag{4.1}$$

where $W_k \in C^\infty(\mathbb{R} \times M \times M, |T(\mathbb{R} \times M \times M)|^{1/2})$ is the Schwartz kernel of the operator $R(k)W$, $\Delta^* : C^\infty(\mathbb{R} \times M \times M, |T(\mathbb{R} \times M \times M)|^{1/2}) \to C^\infty(\mathbb{R}, |T\mathbb{R}|^{1/2}) \otimes C^\infty(M, |TM|)$ is defined by the formula

$$\Delta^*(s_1 \otimes s_2 \otimes s_3)(t,x) = s_1(t) \otimes (s_2(x) \otimes s_3(x)), \quad t \in \mathbb{R}, \quad x \in M,$$

where $s_1 \in C^\infty(\mathbb{R}, |T\mathbb{R}|^{1/2})$, $s_2 \in C^\infty(M, |TM|^{1/2})$, $s_3 \in C^\infty(M, |TM|^{1/2})$, and

$$\pi_* : C^\infty(\mathbb{R}, |T\mathbb{R}|^{1/2}) \otimes C^\infty(M, |TM|) \to C^\infty(\mathbb{R}, |T\mathbb{R}|^{1/2})$$

is given by integration along fibers of the projection π.

It is known that $\pi_*\Delta^* \in I^0(\mathbb{R} \times M \times M \times \mathbb{R}, \Gamma)$, where Γ is the conormal bundle to the diagonal in $\mathbb{R} \times M \times M \times \mathbb{R}$:

$$\Gamma = \{(t, \tau_1, \nu_1, \nu_2, t, \tau_2) \in T^*\mathbb{R} \times T^*M \times T^*M \times T^*\mathbb{R} : \nu_1 = -\nu_2, \tau_1 = -\tau_2\}.$$

There is a commutative diagram

$$
\begin{tikzcd}
\Gamma \arrow[d, "\varphi"'] & \mathcal{Z} \arrow[l, "p_1"'] \arrow[d, "p_2"] \\
T^*(\mathbb{R}\times M\times M) & \mathbb{R}\times G_{\mathcal{F}_N} \arrow[l, "\Pi"']
\end{tikzcd}
\tag{4.2}
$$

where

$$p_1(t,\gamma,\nu) = (\Pi(t,\gamma,\nu), t, -p(\nu)) = (t, p(\nu), \nu, -\nu, t, -p(\nu)), \quad (t,\gamma,\nu)\in \mathcal{Z},$$

p_2 is a natural inclusion and

$$\varphi(t,\tau,\nu,-\nu,t,-\tau) = (t,\tau,\nu,-\nu), \quad (t,\tau,\nu,-\nu,t,-\tau)\in\Gamma.$$

It is easy to see that (4.2) is a fiber product diagram, that is,

$$\mathcal{Z} \cong \{(x,y)\in(\mathbb{R}\times G_{\mathcal{F}_N})\times\Gamma : \Pi(x)=\varphi(y)\}.$$

Using this fact and the functoriality properties of the wave-front sets (see, for instance, [12]), one get immediately the description of the singularities of the distribution θ_k, given by Theorem 3.

To finish the proof of Theorem 6, we will state under the assumption on the flow f_t to be clean in the sense of Definition 5 that $\pi_*\Delta^* W_k$ is a Lagrangian distribution and compute its symbol. We begin with computation of the symbol of the operator $R(k)W$. Recall first the description of the principal symbol of the operator $R(k)$.

According to the short exact sequence

$$0 \to T\mathcal{G}_{\mathcal{F}_N} \to TG_{\mathcal{F}_N} \to H\mathcal{G}_{\mathcal{F}_N} \to 0,$$

the half-density vector bundle on $G_{\mathcal{F}_N}$ can be decomposed as

$$|TG_{\mathcal{F}_N}|^{1/2} \cong |T\mathcal{G}_{\mathcal{F}_N}|^{1/2}\otimes|H\mathcal{G}_{\mathcal{F}_N}|^{1/2},$$

where $|H\mathcal{G}_{\mathcal{F}_N}|^{1/2}$ is the transverse half-density bundle on $G_{\mathcal{F}_N}$: $|H_{(\gamma,\nu)}\mathcal{G}_{\mathcal{F}_N}|^{1/2} \cong |H_\nu\mathcal{F}_N|^{1/2} \cong |H_{dh^*_\gamma(\nu)}\mathcal{F}_N|^{1/2}$.

Let $|dy\wedge d\eta|^{1/2}\in C^\infty(N^*\mathcal{F}, |H\mathcal{F}_N|^{1/2})$ be given by the Liouville form of the canonical transverse symplectic structure on the foliated manifold $(N^*\mathcal{F},\mathcal{F}_N)$, and $r_N^*(|dy\wedge d\eta|^{1/2}) \in C^\infty(G_{\mathcal{F}_N}, |H\mathcal{G}_{\mathcal{F}_N}|^{1/2})$ its pull back via the map $r_N : G_{\mathcal{F}_N}\to N^*\mathcal{F}$.

Recall that the space $S^m(G_{\mathcal{F}_N}, |TG_{\mathcal{F}_N}|^{1/2})$ is defined to be the space of all smooth sections s of the vector bundle $|TG_{\mathcal{F}_N}|^{1/2}$ on $G_{\mathcal{F}_N}$ homogeneous of degree m such that $\pi_G(\mathrm{supp}\, s)$ is compact in $G_{\mathcal{F}}$.

The half-density principal symbol of $R(k)$ is an element of $S^0(G_{\mathcal{F}_N}, |TG_{\mathcal{F}_N}|^{1/2})$ given by the formula

$$\sigma(R(k))(\gamma,\nu) = \pi_G^* k(\gamma,\nu)\otimes r_N^*(|dy\wedge d\eta|^{1/2}), \quad (\gamma,\nu)\in G_{\mathcal{F}_N}.$$

The Maslov bundle $M(\mathbb{R}\times G_{\mathcal{F}_N},\Pi)$ of the immersed canonical relation $(\mathbb{R}\times G_{\mathcal{F}_N},\Pi)$ restricted to $t=0$ is isomorphic to the Maslov bundle $M(G_{\mathcal{F}_N})$ of $G_{\mathcal{F}_N}$, therefore, it has a canonical constant section, which extends to a global section s of $M(\mathbb{R}\times G_{\mathcal{F}_N},\Pi)$ by requiring it to be constant along each bicharacteristic $(t,\tau,\nu_1,\nu_2), \nu_1=f_{-t}(\nu_2), t\in\mathbb{R}$.

Using the description of the principal symbol of the operator $W(t)$ given, for instance, in [3] and the composition theorem of Fourier integral operators, we get immediately that the principal symbol of the operator $R(k)W$ is an element of $S^0(\mathbb{R}\times G_{\mathcal{F}_N}, M(\mathbb{R}\times G_{\mathcal{F}_N},\Pi)\otimes |T(\mathbb{R}\times G_{\mathcal{F}_N})|^{1/2})$, whose value at a point $(t,\gamma,\nu)\in\mathbb{R}\times G_{\mathcal{F}_N}$ is given by

$$\sigma(R(k)W)(t,\gamma,\nu)=e^{i\int_0^t \sigma_{\mathrm{sub}}(P)(f_{-s}dh_\gamma^*(\nu))ds} s\otimes|dt|^{1/2}\otimes\pi_G^*k(\gamma,\nu)\otimes r_N^*(|dy\wedge d\eta|^{1/2}).$$

Now let us turn to the composition (4.1). First, we check the corresponding cleanness assumption.

Lemma 11. *The assumption on the flow f_t to be clean on $\mathcal{Z}_t$ in the sense of Definition 5 guarantees that the composition of $\mathbb{R}\times G_{\mathcal{F}_N}$ with Γ is clean.*

Proof. By definition, the composition of $\mathbb{R}\times G_{\mathcal{F}_N}$ with Γ is clean iff $\mathcal{Z}_t$ is a submanifold of $\mathbb{R}\times G_{\mathcal{F}_N}$ and in addition the fiber product diagram (4.2) is clean at any point $(t,\gamma,\nu)\in\mathcal{Z}$, that is, the linearized diagram

$$\begin{array}{ccc} T_{p_1(t,\gamma,\nu)}\Gamma & \xleftarrow{dp_1} & T_{(t,\gamma,\nu)}\mathcal{Z} \\ {\scriptstyle d\varphi}\big\downarrow & & \big\downarrow{\scriptstyle dp_2} \\ T_{(t,p(\nu),\nu,-\nu)}(T^*(\mathbb{R}\times M\times M)) & \xleftarrow{d\Pi} & T_{(t,\gamma,\nu)}(\mathbb{R}\times G_{\mathcal{F}_N}) \end{array} \tag{4.3}$$

is a fiber product diagram. Since $T_{(t,\gamma,\nu)}\mathcal{Z}$ is always contained in $T_\nu N^*\mathcal{F}\oplus T_{dh_\gamma^*(\nu)}N^*\mathcal{F}$, this is true iff the diagram

$$\begin{array}{ccc} T_{p_1(t,\gamma,\nu)}(\Gamma\bigcap T^*\mathbb{R}\times N^*\mathcal{F}\times N^*\mathcal{F}\times T^*\mathbb{R}) & \xleftarrow{dp_1} & T_{(t,\gamma,\nu)}\mathcal{Z} \\ {\scriptstyle d\varphi}\big\downarrow & & \big\downarrow{\scriptstyle dp_2} \\ T_{(t,p(\nu),\nu,-\nu)}(T^*\mathbb{R}\times N^*\mathcal{F}\times N^*\mathcal{F}) & \xleftarrow{d\Pi} & T_{(t,\gamma,\nu)}(\mathbb{R}\times G_{\mathcal{F}_N}) \end{array} \tag{4.4}$$

is a fiber product diagram.

The diagram (4.4) has a subdiagram

$$\begin{array}{ccc} L_{p_1(t,\gamma,\nu)}\Gamma & \xleftarrow{dp_1} & L_{(t,\gamma,\nu)}\mathcal{Z} \\ {\scriptstyle d\varphi}\big\downarrow & & \big\downarrow{\scriptstyle dp_2} \\ 0\oplus T_\nu\mathcal{F}_N\oplus T_{-\nu}\mathcal{F}_N & \xleftarrow{d\Pi} & L_{(t,\gamma,\nu)}(\mathbb{R}\times G_{\mathcal{F}_N}) \end{array} \tag{4.5}$$

where $L_{(t,\gamma,\nu)}(\mathbb{R}\times G_{\mathcal{F}_N})\subset T_{(t,\gamma,\nu)}(\mathbb{R}\times G_{\mathcal{F}_N})$ is given by

$$\begin{aligned} &L_{(t,\gamma,\nu)}(\mathbb{R}\times G_{\mathcal{F}_N})\\ &\qquad=\{(U,V_1,V_2,W)\in T_t(\mathbb{R})\oplus T_\nu\mathcal{F}_N\oplus T_{dh_\gamma^*(\nu)}\mathcal{F}_N\oplus H_{(\gamma,\nu)}\mathcal{G}_{\mathcal{F}_N} : U=0, W=0\}\\ &\qquad\qquad\cong T_\nu\mathcal{F}_N\oplus T_{dh_\gamma^*(\nu)}\mathcal{F}_N \end{aligned}$$

$L_{(t,\gamma,\nu)}\mathcal{Z} \subset T_{(t,\gamma,\nu)}\mathcal{Z}$ by

$$L_{(t,\gamma,\nu)}\mathcal{Z} = \{(U, V_1, V_2, W) \in T_t(\mathbb{R}) \oplus T_\nu\mathcal{F}_N \oplus T_{dh^*_\gamma(\nu)}\mathcal{F}_N \oplus H_{(\gamma,\nu)}\mathcal{G}_{\mathcal{F}_N} \\ : U = 0, V_1 = -df_{-t}(dh^*_\gamma(\nu))(V_2), W = 0\} \cong T_\nu\mathcal{F}_N$$

and $L_{p_1(t,\gamma,\nu)}\Gamma \subset T_{p_1(t,\gamma,\nu)}\Gamma$ by

$$L_{p_1(t,\gamma,\nu)}\Gamma \cong \{(U_1, V_1, V_2, U_2) \in T_{(t,p(\nu))}(T^*\mathbb{R}) \oplus T_\nu\mathcal{F}_N \oplus T_{-\nu}\mathcal{F}_N \oplus T_{(t,-p(\nu))}(T^*\mathbb{R}) \\ : U_1 = U_2 = 0, V_1 = -V_2\}$$

which can be easily seen to be a fiber diagram.

Therefore, the diagram (4.4) is a fiber product diagram iff the quotient diagram is a fiber product diagram:

$$\begin{tikzcd} H_{p_1(t,\gamma,\nu)}\Gamma \arrow[d, "d\varphi"'] & H_{(t,\gamma,\nu)}\mathcal{Z} \arrow[l, "dp_1"'] \arrow[d, "dp_2"'] \\ T_{(t,p(\nu))}(T^*\mathbb{R}) \oplus H_\nu\mathcal{F}_N \oplus H_{-\nu}\mathcal{F}_N & H_{(t,\gamma,\nu)}(\mathbb{R} \times G_{\mathcal{F}_N}) \arrow[l, "d\Pi"'] \end{tikzcd} \tag{4.6}$$

where

$$H_{(t,\gamma,\nu)}(\mathbb{R} \times G_{\mathcal{F}_N}) \\ = \{(U, V_1, V_2, W) \in T_t(\mathbb{R}) \oplus T_\nu\mathcal{F}_N \oplus T_{dh^*_\gamma(\nu)}\mathcal{F}_N \oplus H_{(\gamma,\nu)}\mathcal{G}_{\mathcal{F}_N} : V_1 = 0, V_2 = 0\} \\ \cong T_t(\mathbb{R}) \oplus H_{(\gamma,\nu)}\mathcal{G}_{\mathcal{F}_N},$$

$$H_{(t,\gamma,\nu)}\mathcal{Z} = T_{(t,\gamma,\nu)}\mathcal{Z}/L_{(t,\gamma,\nu)}\mathcal{Z} \cong T_\nu Z_t/T_\nu\mathcal{F}_N,$$

$$H_{p_1(t,\gamma,\nu)}\Gamma \cong \{(U_1, V_1, V_2, U_2) \in T_{(t,p(\nu))}(T^*\mathbb{R}) \oplus H_\nu\mathcal{F}_N \oplus H_{-\nu}\mathcal{F}_N \oplus T_{(t,-p(\nu))}(T^*\mathbb{R}) \\ : U_1 = -U_2, V_1 = -V_2\}.$$

In its turn, the diagram (4.6) is a fiber product diagram iff the flow f_t is clean on $\mathcal{Z}_t$ in the sense of Definition 5. □

For any connected component $\mathcal{Z}_j$ of $\mathcal{Z}_t$, the excess of the clean diagram (4.2) equals d_j, the dimension of the relative fixed point set $S\mathcal{Z}_j$ in $G_{SN^*\mathcal{F}}$. By the composition theorem of Fourier integral operators, θ_k belongs to $\bigoplus_j I^{\frac{d_j-p}{2}-\frac{1}{4}}(\Lambda_t)$, where $\Lambda_t = \{(t,\tau) \in T^*\mathbb{R} : \tau \in \mathbb{R}_-\}$, that proves the desired representation of θ_k in the form (2.5) and the existence of the asymptotic expansion (2.6) for $\alpha_j(s,k)$.[1]

To obtain the explicit formula for the leading coefficients $\alpha_{j,0}$, we compute the principal symbol of θ_k, $\sigma(\theta_k)$, following the arguments in [13].

[1]Note that the appearance of the term $-p/2$ in the exponent is due to the fact $R(k)W \in I^{-p/2-1/4}(M \times M; \mathbb{R} \times G_{\mathcal{F}_N}, \Pi)$.

Fix a connected component $\mathcal{Z}_j$ of $\mathcal{Z}_t$ and $(t,\gamma,\nu)\in\mathcal{Z}_j$. The fiber product diagram (4.3) defines a composition map [3, 11]

$$*: |T_{p_1(t,\gamma,\nu)}\Gamma|^{1/2}\otimes|T_{(t,\gamma,\nu)}(\mathbb{R}\times G_{\mathcal{F}_N})|^{1/2}\to|T_{(t,-p(\nu))}(\Lambda_t)|^{1/2}\otimes|T_{(t,\gamma,\nu)}\mathcal{Z}_j| \tag{4.7}$$

and, due to (4.1) and the composition theorem for Fourier integral operators, the value of the principal symbol $\sigma(\theta_k)\in|T\Lambda_t|^{1/2}$ at a point $(t,\tau)\in\Lambda_t$ is given by integration over $\mathcal{Z}_j$ of $\sigma(\pi_*\Delta^*)*\sigma(W_k)\in C^\infty(\Lambda_t\times\mathcal{Z}_j,|T\Lambda_t|^{1/2}\otimes|T\mathcal{Z}_j|)$.

Using the functoriality of the $*$ operation on half densities with respect to reduction (cf. [13, proof of Lemma 4.6]), the computation of the $*$-product $\sigma(\pi_*\Delta^*)*\sigma(W_k)$ can be reduced to the computation of a $*$-product defined by the transverse fiber product diagram (4.6):

$$*_t: |H_{p_1(t,\gamma,\nu)}\Gamma|^{1/2}\otimes|H_{(t,\gamma,\nu)}(\mathbb{R}\times G_{\mathcal{F}_N})|^{1/2}\to|T_{(t,-p(\nu))}\Lambda_t|^{1/2}\otimes|H_{(t,\gamma,\nu)}\mathcal{Z}_j|.$$

More precisely, we apply the result stated in [13, proof of Lemma 4.6] with a symplectic vector space $\mathcal{V}=T_{(t,p(\nu),\nu,-\nu,t,-p(\nu))}(T^*\mathbb{R}\times T^*M\times T^*M\times T^*\mathbb{R})$, two Lagrangian subspaces in $\mathcal{V}$: $\Lambda_1=T_{p_1(t,\gamma,\nu)}\Gamma$ and Λ_2, which is the image of $T_{(t,\gamma,\nu)}(\mathbb{R}\times G_{\mathcal{F}_N})$ in $\mathcal{V}$ and the reduction given by the coisotropic subspace $\Gamma=T_{(t,p(\nu),\nu,-\nu,t,-p(\nu))}(T^*\mathbb{R}\times N^*\mathcal{F}\times N^*\mathcal{F}\times T^*\mathbb{R})$.

By this result, the leafwise component of $\sigma(\theta_k)$, $\sigma_l(\theta_k)\in|L_{(t,\gamma,\nu)}\mathcal{Z}_j|$, is obtained from the leafwise component of $\sigma(R(k)W)$:

$$\sigma_l(R(k)W)=e^{i\int_0^t\sigma_{\mathrm{sub}}(P)(f_{-s}dh^*_\gamma(\nu))ds}\pi^*_Gk$$

by application of the restriction map $R_{\mathcal{Z}}$:

$$\sigma_l(\theta_k)=e^{i\int_0^t\sigma_{\mathrm{sub}}(P)(f_{-s}dh^*_\gamma(\nu))ds}R_{\mathcal{Z}}\pi^*_Gk,$$

and the transversal component of $\sigma(\theta_k)$, $\sigma_t(\theta_k)\in|T_{(t,-p(\nu))}(\Lambda_t)|^{1/2}\otimes|H_{(t,\gamma,\nu)}\mathcal{Z}_j|$, is equal to the $*_t$-product of the transverse component of $\sigma(R(k)W)$,

$$\sigma_t(R(k)W)=|dt|^{1/2}\otimes r_N^*(|dy\wedge d\eta|^{1/2}),$$

and the transverse component of $\sigma(\pi_*\Delta^*)$. By [3], we get

$$\sigma_t(\theta_k)=|dt|^{1/2}\otimes d\mu_{\mathcal{Z}_j}$$

that completes the calculation of the half-density principal symbol of θ_k and implies the formula (2.7) for the leading coefficients $\alpha_{j,0}$ as in [3].

5 Maslov indices

In this section, we define the Maslov factors σ, corresponding to $(\gamma,\nu)\in\mathcal{Z}_t$. For this goal, we will use local coordinates on the holonomy groupoid G described, for instance, in [8, 4]. Choose a pair of compatible foliated charts near the points $\pi(\nu)$ and $\pi(f_t(\nu))$ with the coordinates (x,y) and (x',y), corresponding to $\gamma\in G$. Then we have the corresponding coordinates in $H_\nu\mathcal{F}_N=T_\nu N^*\mathcal{F}/T_\nu\mathcal{F}_N$ defined as $(\delta y,\delta\eta)$, and the vertical and horizontal

subspaces V_ν and H_ν, given by the equations $\delta y = 0$ and $\delta\eta = 0$ accordingly. The linear holonomy map $dH_{(\gamma,\nu)}$ of the foliation $(N^*\mathcal{F}, \mathcal{F}_N)$ defines an isomorphism of the symplectic spaces $H_{f_t(\nu)}\mathcal{F}_N$ and $H_\nu\mathcal{F}_N$, which preserves the vertical and horizontal subspaces. Due to this isomorphism, we can obtain a closed curve $\omega_{(\gamma,\nu)}$ in the Lagrangian Grassmannian $\mathcal{G}$ of the symplectic space $H_\nu\mathcal{F}_N$, pulling back, via df_t, the vertical subspace at $f_t(\nu)$ for t between 0 and T. Denote by $\kappa_{(\gamma,\nu)}$ the intersection number of ω with the horizontal subspace H_ν:

$$\kappa_{(\gamma,\nu)} = [\omega_{(\gamma,\nu)} : H_\nu].$$

Let $\chi(t, x, y, \xi, \eta)$ be the generating function of the canonical transformation f_t in the chosen coordinates. Recall that χ is the solution of the Cauchy problem

$$d_t\chi = p(x, y, d_x\chi, d_y\chi), \quad \chi(0, x, y, \xi, \eta) = x\xi + y\eta. \tag{5.1}$$

By the holonomy invariance of p, it can be easily seen that $\chi(t, x, y, 0, \eta)$ is independent of x and ξ: $\chi(t, x, y, 0, \eta) = \chi(t, y, \eta)$. Let

$$R_{(\gamma,\nu)} = \begin{bmatrix} d^2_{yy}\chi & d^2_{y\eta}\chi & -1 \\ d^2_{\eta y}\chi & d^2_{\eta\eta}\chi & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

We define the Maslov factor $\sigma(\gamma, \nu)$ as

$$\sigma(\gamma, \nu) = \operatorname{sgn} R_{(\gamma,\nu)} + 2\kappa_{(\gamma,\nu)}, \quad (\gamma, \nu) \in \mathcal{Z}.$$

It is clear that $\sigma(\gamma, \nu)$ is a locally constant function on $\mathcal{Z}$.

To handle with Maslov factors in the proof of Theorem 6, let us write the Schwartz kernel of the operator $R(k)$ in a foliated coordinate chart on G given by a pair of compatible coordinate systems as $k(x, x_1, y)\delta(y - y_1)$, and the Schwartz kernel $W(t)$ microlocally as an oscillatory integral of the form

$$\int e^{i\alpha(t,x_1,y_1,x_2,y_2,\xi,\eta)} a(t, x_1, y_1, x_2, y_2, \xi, \eta)\, d\xi d\eta,$$

where

$$\alpha(t, x_1, y_1, x_2, y_2, \xi, \eta) = \chi(t, x_1, y_1, \xi, \eta) - x_2\xi - y_2\eta$$

and $\chi(t, x_1, y_1, \xi, \eta)$ is the generating function of the canonical transformation f_t given by (5.1). Then the Schwartz kernel of the operator $R(k)W$ is given by the formula

$$\int e^{i\alpha(t,x_1,y_1,x_2,y_2,\xi,\eta)} k(x, x_1, y_1) a(t, x_1, y_1, x_2, y_2, \xi, \eta)\, dx_1 d\xi d\eta,$$

from where one can easily derive the desired assertion, following the arguments of [3].

References

[1] Y. Colin de Verdière. Spectre du Laplacien et longeurs des géodésiques periodiques. *Comp. Math.*, 27:159–184, 1973.

[2] J. Chazarain. Formule de Poisson pour le variétés riemanniennes. *Invent. Math.*, 24:65–82, 1974.

[3] J.J. Duistermaat and V. Guillemin. The spectrum of positive elliptic operators and periodic bicharacteristics. *Invent.Math.*, 29:39–79, 1975.

[4] Yu. A. Kordyukov. Noncommutative spectral geometry of Riemannian foliations. *Manuscripta Math.*, 94:45–73, 1997.

[5] A. Connes. Trace formula in noncommutative geometry and the zeros of the Riemann zeta function. *Selecta Math. (N.S.)*, 5:29–106, 1999.

[6] F. Golse and E. Leichtnam. Applications of Connes' geodesic flow to trace formulas in noncommutative geometry. *J. Funct. Anal.*, 160:408–436, 1998.

[7] B. L. Reinhart. *Differential Geometry of Foliations*. Springer, Berlin Heidelberg New York, 1983.

[8] A. Connes. Sur la théorie non-commutative de l'intégration. In *Algèbres d'opérateurs*, volume 725 of *Lecture Notes Math.*, pages 19–143. Springer, Berlin Heidelberg New York, 1979.

[9] V. Guillemin and S. Sternberg. Some problems in integral geometry and some related problems in microlocal analysis. *Amer. J. Math.*, 101:915–959, 1979.

[10] Yu. A. Kordyukov. The transversal wave equation and the noncommutative geodesic flow in Riemannian foliations. Preprint ETH Zürich, 1997; dg-ga/9703015.

[11] L. Hörmander. *The analysis of linear partial differential operators IV*. Springer, Berlin Heidelberg New York, 1986.

[12] V. Guillemin and S. Sternberg. *Geometric Asymptotics*. American Mathematical Society, Providence, R. I., 1977.

[13] V. Guillemin and A. Uribe. Circular symmetry and the trace formula. *Invent. Math.*, 96:385–423, 1989.